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Shunsuke Takagi

Cold Regions Research and Engineering Laboratory
Hanover, New Hampshire

September 1972

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Final Report
ARO-Durham Contract I-466-E

September 1972

U.S. ARMY COLD REGIONS RESEARCH AND ENGINEERING LABORATORY
HANOVER, NEW HAMPSHIRE 03755

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

U.S. Army Cold Regions Research & Engineering Lab
Hanover, New Hampshire 03755

2a. REPORT SECURITY CLASSIFICATION Unclassified

2b. GROUP NA

3. REPORT TITLE

Analysis of the Buckling of an Elastic Plate Floating on Water and Stressed Uniformly Along the Periphery of an Internal Hole

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Final Report: 1 March 1972 - 30 June 1972

5. AUTHOR(S) (First name, middle initial, last name)

Shunsuke Takagi

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1972	30p	5

8a. CONTRACT OR GRANT NO.

b. PROJECT NO. 20061102B14C

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S) NA

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) I-466.1-E

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES

None

12. SPONSORING MILITARY ACTIVITY

U. S. Army Research Office-Durham
Box CM, Duke Station
Durham, North Carolina 27706

13. ABSTRACT

This problem can be reduced to an eigenvalue problem of a fourth order ordinary differential equation in a semi-infinite region. The series that satisified the boundary condition at infinity is asymptotic, which cannot satisfy the boundary condition at a finite point and fails to yield the eigenvalues. Theoretically this difficulty can be overcome, as explained in this report, by converting the divergent asymptotic series to a convergent factorial series.
The Dartmouth Time Sharing System (DTSS) was used for the numerical computation of the factorial series. It was found that some of the factorial series converge very slowly and could not yield any meaningful result even in the utmost capability of the DTSS. It was decided that the use of the factorial series should abandoned.
The reason for the failure is described.

14. KEY WORDS

Differential equations
Eigenvalues
Asymptotic series
Factorial series
Buckling
Plates (structural forms)
Stress concentration

DD FORM 1473, 1 NOV 65 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

Unclassified
Security Classification

1. Problem to be solved

We consider a thin plate floating on water stressed with uniform horizontal pressure along the periphery of an internal hole. We are interested in formulating the buckling pressure and the location of the maximum deflection in terms of the radius of the hole.

The vertical deflection of an elastic plate, which rests on a liquid and is subjected to a vertical load q and the horizontal internal stress of components N_{xx}, N_{yy}, and N_{xy}, is described by the partial differential equation

$$D\nabla^4 w + \gamma w = q + N_{xx}\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_{yy}\frac{\partial^2 w}{\partial y^2} \qquad (1)$$

where D is the flexural rigidity, γ is the specific weight of the liquid (ref. 1).

In our problem, the deformation is cylindrically symmetric around the axis at the center of the circular hole. Let r be the radial distance from the center of the hole. Then (1) becomes

$$\ell_o^4 \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) w + w = \frac{1}{\gamma}\left\{N_{rr}\frac{d^2 w}{dr^2} + N_{\theta\theta}\frac{1}{r}\frac{dw}{dr}\right\}, \qquad (2)$$

where ℓ_o is the characteristic length, and N_{rr} and $N_{\theta\theta}$ are the radial, and hoop horizontal stresses in the plate. In the general polar coordinates

$$N_{xx}\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_{yy}\frac{\partial^2 w}{\partial y^2}$$

$$= N_{rr}\frac{\partial^2 w}{\partial r^2} + 2N_{r\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) + N_{\theta\theta}\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right).$$

Mansfield (ref. 1) assumes that the horizontal stress N_{xx}, N_{xy}, N_{yy} is in equilibrium by itself and shows that they are derived from a biharmonic function φ by

$$N_{xx} = \frac{\partial^2 \varphi}{\partial y^2}$$

$$N_{yy} = \frac{\partial^2 \varphi}{\partial x^2}$$

$$N_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y}$$

In the general polar coordinates they are:

$$N_{rr} = \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}$$

$$N_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right)$$

$$N_{\theta\theta} = \frac{\partial^2 \varphi}{\partial r^2}$$

In our problem φ is a function of r only and must tend to zero when r becomes infinite. Then we must have

$$N_{rr} = -\frac{A}{r^2},$$

$$N_{r\theta} = 0,$$

and

$$N_{\theta\theta} = \frac{A}{r^2},$$

where A must be positive because N_{rr} is pressure. We express them as

$$N_{rr} = -\frac{a\gamma \ell_o^4}{r^2}$$

and (3)

$$N_{\theta\theta} = \frac{a\gamma \ell_o^4}{r^2}$$

where a is a positive constant to be determined.

Introudce the non-dimensional length x,

$$x = r/\ell_o \tag{4}$$

Then (2) becomes

$$\frac{d^4w}{dx^4} + \frac{2}{x}\frac{d^3w}{dx^3} - \frac{1-a}{x^2}\frac{d^2w}{dx^2} + \frac{1-a}{x^3}\frac{dw}{dx} + w = 0 \tag{5}$$

The boundary conditions we presently choose for this problem are:

$$\left.\begin{aligned} w &= 0 \\ \frac{dw}{dx} &= 0 \end{aligned}\right\} \text{at } x = x_o$$

$$\left.\begin{aligned} w &= 0 \\ \frac{dw}{dx} &= 0 \end{aligned}\right\} \text{at } x = \infty$$

where x_o is the value of x at the periphery of the hole.

2. Asymptotic Power Series Solution

Wassow (ref. 2) describes a matrix theory of ordinary differential equations. According to his theory the differential equation (5) has an asymptotic power series solution of the form

$$w = e^{-\lambda x} \sum_{n=0}^{\infty} p_n \, x^{\mu - n} \tag{6}$$

where p_n, λ and μ are constants. We will evaluate the constants. Wassow's theory, which are concerned with proving the existence of the asymptotic power series solution, is not convenient to this purpose. The constants will be determined here by directly substituting series (6) into equation (5). Wassow's theory, however, is needed for explaining the existence of the factorial series solution, and will be outlined in the next section.

The result of the substitution of (6) into (5) can be rearranged as:

$$\sum_{n=0}^{\infty} A_n \, x^{\mu - n} = 0$$

From $A_o = 0$ we find

$$\lambda^4 + 1 = 0$$

From $A_1 = 0$ we find

$$\mu = - \frac{1}{2}$$

For simplicity we put

$$p_o = 1. \tag{7}$$

Constants p_1 and p_2 are found, from $A_2 = 0$ and $A_3 = 0$, as

$$p_1\lambda = -\frac{1}{4}\left(a + \frac{1}{2}\right) \tag{8}$$

$$p_2\lambda^2 = \frac{1}{32}\left(a + \frac{1}{2}\right)\left(a + \frac{9}{2}\right), \tag{9}$$

respectively. For p_n $(n \geqq 3)$ we can find from $A_n = 0$ $(n \geqq 4)$ the recurrence formula,

$$\begin{aligned} p_n\lambda^n = -\frac{1}{4n} \Big\{ & \frac{1}{2}\left(-2(1-a) + 3(2n-1)^2\right) p_{n-1}\lambda^{n-1} + \\ & + \left(-2(1-a)(n-1) + (2n-1)(n-1)(2n-3)\right) p_{n-2}\lambda^{n-2} + \\ & + \frac{1}{16}(2n-1)(2n-5)\left(4a + (2n-1)(2n-5)\right) p_{n-3}\lambda^{n-3}\Big\}. \end{aligned} \tag{10}$$

Equation (10) enables us to determine $p_n\lambda^n$ $(n \geq 3)$ as a polynomial of constant a using (7), (8), and (9) as the initial conditions. To find the polynomial let

$$p_n\lambda^n = \sum_{r=0}^{n} \beta_r^n a^r \tag{11}$$

From the conditions (7), (8), and (9) we find

$$\beta_o^o = 1$$

$$\beta_o^1 = -\frac{1}{8}, \quad \beta_1^1 = -\frac{1}{4}$$

$$\beta_o^2 = \frac{9}{128}, \quad \beta_1^2 = \frac{5}{32}, \quad \beta_2^2 = \frac{1}{32}$$

Substituting (11) into (10) we find the recurrence formulas,

$$-4n\beta_n^n = \beta_{n-1}^{n-1}$$

$$-4n\beta_{n-1}^n = \beta_{n-2}^{n-1} + f(n)\,\beta_{n-1}^{n-1} + 2(n-1)\,\beta_{n-2}^{n-2}$$

$$-4n\beta_{n-2}^n = \beta_{n-3}^{n-1} + f(n)\,\beta_{n-2}^{n-1} + 2(n-1)\,\beta_{n-3}^{n-2} + g(n)\,\beta_{n-2}^{n-2} +$$

$$+ h(n)\,\beta_{n-3}^{n-3}$$

$$-4n\beta_r^n = \beta_{r-1}^{n-1} + f(n)\,\beta_r^{n-1} + 2(n-1)\,\beta_{r-1}^{n-2} + g(n)\,\beta_r^{n-2} +$$

$$+ h(n)\,\beta_{r-1}^{n-3} + h^2(n)\,\beta_r^{n-3} \quad (\text{for } n-3 \geqq r \geqq 1)$$

$$-4n\beta_o^n = f(n)\,\beta_o^{n-1} + g(n)\,\beta_o^{n-2} + h^2(n)\,\beta_o^{n-3}$$

where

$$f(n) = 6(n-\tfrac{1}{2})^2 - 1$$

$$g(n) = (n-1)\,(4(n-1)^2 - 3)$$

$$h(n) = (n-\tfrac{1}{2})\,(n-\tfrac{5}{2})$$

For $n = 3$ the fourth formula is not needed. For $n > 3$ the fourth formula must be used with $r = n-3, n-2 \ldots, 1$.

Of the four roots of $\lambda^4 = -1$, we choose

$$\lambda = (1 \pm i)/\sqrt{2}$$

to satisfy the boundary condition of w at $x = \infty$.

The solution converging at $x = \infty$ is thus:

$$w = Ce^{-x/\sqrt{2}} \sum_{n=o}^{\infty} x^{-n-1/2} \exp\left(\pm i\left(\frac{x}{\sqrt{2}} + \frac{n\pi}{4}\right)\right) \sum_{r=o}^{n} \beta_r^n a^r, \tag{12}$$

where C is a complex constant. The real value solution will be expressed as

$$w = Aw_1 + Bw_2 \tag{13}$$

with real constants A and B, where

$$w_1 = e^{-x/\sqrt{2}} \sum_{n=o}^{\infty} x^{-n-1/2} \cos\left(\frac{x}{\sqrt{2}} + \frac{n\pi}{4}\right) \sum_{r=o}^{n} \beta_r^n a^r$$

and (14)

$$w_2 = e^{-x/\sqrt{2}} \sum_{n=o}^{\infty} x^{-n-1/2} \sin\left(\frac{x}{\sqrt{2}} + \frac{n\pi}{4}\right) \sum_{r=o}^{n} \beta_r^n a^r$$

Or changing the order of summations,

$$w_1 = e^{-x/\sqrt{2}} \sum_{r=o}^{\infty} a^r \sum_{n=r}^{\infty} \beta_r^n x^{-n-1/2} \cos\left(\frac{x}{\sqrt{2}} + \frac{n\pi}{4}\right)$$

and (15)

$$w_2 = e^{-x/\sqrt{2}} \sum_{r=o}^{\infty} a^r \sum_{n=r}^{\omega} \beta_r^n x^{-n-1/2} \sin\left(\frac{x}{\sqrt{2}} + \frac{n\pi}{4}\right)$$

The last form of w_1 and w_2 are needed in the next section for evaluating constant a.

According to Wassow (ref. 2, Theor. 12.3) the series

$$\sum_{n=o}^{\infty} x^{-n-1/2}\left(A\cos\left(\frac{x}{\sqrt{2}}+\frac{n\pi}{4}\right)+B\sin\left(\frac{x}{\sqrt{2}}+\frac{n\pi}{4}\right)\right)\sum_{r=o}^{n} \beta_r^n a^r$$

is asymptotic. Still for this case (15) is valid, because the order of summations in an asymptotic power series can be exchanged (ref. 2, Theor. 9.5).

The asymptotic power series (15), which diverges when n is increased indefinitely for a fixed x, is not convenient for determining constant a and the ratio A:B from the conditions for at $x = x_o$. The computation of the asymptotic power series can be avoided, as explained below, because the asymptotic power series can be transformed to a convergent factorial series.

3. Existence of a Convergent Factorial Series Solution

A factorial series is a series of the form

$$\sum_{r=o}^{\infty} \frac{a_r}{x(x+1)\ \ldots\ (x+r)}$$

According to Wassow (ref. 2, Theor. 47.1), the differential equation (5) has a convergent factorial series solution. For the proof he uses a matrix expression of ordinary differential equations. The differential equation (5) will be given a matrix expression in this section in order to show that our equation satisfies the requirements in Wassow's theorem.

Let

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

be a four-dimensional vector defined as

$$y_1 = w$$

$$y_2 = \frac{dw}{dx}$$

$$y_3 = \frac{d^2w}{dx^2}$$

and

$$y_4 = \frac{d^3w}{dx^3}$$

Then

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = y_3$$

$$\frac{dy_3}{dx} = y_4$$

and

$$\frac{dy_4}{dx} = -\frac{2}{x}\ y_4 + \frac{1-a}{x^2}\ y_3 - \frac{1-a}{x^3}\ y_2 - y_1$$

The last four equations may be lumped into a matrix equation,

$$\frac{dY}{dx} = A(x)\; Y \tag{16}$$

where

$$A(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 - \frac{1-a}{x^3} & \frac{1-a}{x^2} & & -\frac{2}{x} \end{bmatrix} \tag{17}$$

The entries of the matrix function $A(x)$ are finite at $x = \infty$ and analytic in the neighborhood of $x = \infty$, satisfying a condition in Wassow's theorem.

The eigenvalues of the matrix $A(\infty)$ are the roots of the determinant equation,

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ -1 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

i.e.

$$\lambda^4 + 1 = 0.$$

The eigenvalues are therefore distinct. Thus our equation satisfies all the requirements of Wassow's theorem.

The resemblance of the matrix equation (16) to an equation of single unknown seems to suggest a similar treatment. Write

$$A(x) = A_0 + A_1x^{-1} + A_2x^{-2} + A_3x^{-3},$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-a & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -(1-a) & 0 & 0 \end{bmatrix}$$

The form of $A(x)$ suggests to integrate (16) to:

$$Y = \exp\,[A_o x + A_1 \log x - A_2 x^{-1} - \frac{1}{2} A_3 x^{-2}] \tag{18}$$

The matrix theory defines the matrix power of e as

$$e^A = \sum_{r=o}^{\infty} \frac{1}{r!} A^r$$

However the relation

$$e^{A+B} = e^A e^B$$

is not true unless

$$AB = BA$$

where A and B are matrices. Because of this restriction, (18) cannot be reduced to a meaningful form. The avoidance of the use of (18) seems to me the key to the theory described by Wassow for proving the existence of the solution in the form of (6).

4. Various Expressions of x^{-p}

The convergent factorial series solution of the differential equation (5) can be found by substituting the factorial series expressions of x^{-p} $(p = 1, 2, 3, \ldots)$ into the asymptotic power series solution (12). The existence theorem (ref. 2, Theor. 47.1) guarantees the validity of the procedure.

In this section, following the traditional treatment, various forms of factorial series expressing x^{-p} $(p = 1, 2, \ldots)$ will be derived. In the numerical computation all forms of the factorial series developed here were tested but none could yield satisfactory results in all the computations.

The proof starts from recalling the definition of the gamma function,

$$\Gamma(p) = \int_0^\infty e^{-\xi}\, \xi^{p-1}\, d\xi,$$

defined for $p > 0$. Changing the variable ξ to t by $\xi = xt$, we find

$$x^{-p} = \frac{1}{\Gamma(p)} \int_0^\infty e^{-xt}\, t^{p-1}\, dt$$

Changing the variable t to z by $z = e^{-t}$, we have

$$x^{-p} = \frac{(-1)^{p-1}}{\Gamma(p)} \int_0^1 z^{x-1} (\log z)^{p-1}\, dz \tag{19}$$

Let

$$\varphi(z) = \frac{(-1)^{p-1}}{\Gamma(p)} (\log z)^{p-1} \tag{20}$$

If we substitute the Taylor series of $\varphi(z)$ developed around $z = 1$,

$$\varphi(z) = \sum_{n=o}^{\infty} \frac{1}{n!} \varphi^{(n)}(1)\ (z-1)^n,$$

into (19) we find

$$x^{-p} = \sum_{n=o}^{\infty} \frac{(-1)^n}{n!} \varphi^{(n)}(1)\ B(x, n+1), \tag{21}$$

where $B(x, n+1)$ is the beta function

$$B(x, n+1) = \int_o^1 z^{x-1}\ (1-z)^n\ dz \tag{22}$$

Various factorial series expressions of x^{-p} stem from (21) as shown below.

A sophisticated device is necessary to derive the successive derivatives $\varphi^{(n)}(1)$: Let $G(\log z)$ be an analytic function of $\log z$. One sees easily that

$$\frac{d^n}{dz^n} G(\log z) = z^{-n} \sum_{\nu=o}^{n-1} (-1)^{\nu}\ S_{\nu}^{n}\ G^{(n-\nu)}\ (\log z) \tag{23}$$

for $n \geqq 1$ with certain constants S_{ν}^{n} which are the same for all functions $G(\xi)$, where

$$G^{(k)}\ (\log z) = \left[\frac{d^k}{d\xi^k}\ G(\xi)\right]_{\xi\ =\ \log\ z}$$

Let $G(\xi) = e^{-c\xi}$ to determine S_{ν}^{n}, where c is a constant. Then $G(\log z) = Z^{-c}$ and $d^n/dz^n\ G(\log z) = (-1)^n\ c(c+1)\ldots(c+n-1)z^{-c-n}$, $G^{(k)}\ (\log z) = (-1)^k c^k z^{-c}$.

Hence (23) reduces to

$$c(c+1)\ldots(c+n-1) = \sum_{\nu=0}^{n-1} S_\nu^n \, c^{n-\nu} \tag{24}$$

For n = 1 and 2, (24) easily yields

$$S_0^1 = 1, \quad S_1^1 = 0$$

$$S_0^2 = 1, \quad S_1^2 = 1, \quad S_2^2 = 0$$

For a large n, recurrence formulas for computing S_ν^n will be found. Increasing n in (24) by 1 yields

$$c(c+1)\ldots(c+n) = \sum_{\nu=0}^{n} S_\nu^{n+1} \quad c^{n+1-\nu}$$

Comparing the two formulas we find the relation,

$$\sum_{\nu=0}^{n} S_\nu^{n+1} \quad c^{n+1-\nu} = (c+n) \sum_{\nu=0}^{n-1} \quad S_\nu^n \quad c^{n-\nu}$$

Because c is arbitrary, the following must hold:

$$S_0^{n+1} = 1,$$

$$S_\nu^{n+1} = S_\nu^n + n\, S_{\nu-1}^n, \text{ for } 1 \leqq \nu \leqq n-1$$

$$S_n^{n+1} = nS_{n-1}^n = n!.$$

and

$$S_{n+1}^{n+1} = 0$$

Because $S_n^n = 0$ for $n \geqq 1$, (24) may be written as

$$c(c+1)\ldots(c+n-1) = \sum_{\nu=o}^{n} S_\nu^n \, c^{n-\nu} \tag{25}$$

in which we may let n = 0 by writing the left-hand side as $\Gamma(c+n)/\Gamma(c)$. Thus we find

$$S_0^0 = 1$$

The numbers S_ν^n are a kind of Stirling numbers.

Stirling number S_ν^n for a fixed n increases rapidly with the increase of ν and suddenly drops to zero at $\nu = n$. The large variation of the values of S_ν^n was one of the causes of the unsatisfactory results in the numerical computation.

Letting $G(\xi) = (-1)^{p-1} \xi^{p-1}/\Gamma(p)$ in (23), we find for a positive integer p

$$\frac{d^n\varphi(z)}{dz^n} = z^{-n} \sum_{\nu=n+1-p}^{n-1} (-1)^\nu \, S_\nu^n \, \frac{(-1)^{p-1}}{\Gamma(p)} \, (p-1)\ldots(p-1-n+\nu+1) \, (\log z)^{p-1-n+\nu}$$

Thus we find

$$\varphi^{(n)}(1) = (-1)^n \, S_{n+1-p}^n$$

Therefore (21) becomes

$$x^{-p} = \sum_{n=o}^{\infty} \frac{1}{n!} \, S_{n+1-p}^n \, B(x, n+1)$$

This reduces, because $S_\nu^n = 0$ for $\nu < 0$, to

$$x^{-p} = \sum_{n=p-1}^{\infty} \frac{1}{n!} S^{n}_{n+1-p} \quad B(x, n+1). \tag{26}$$

Substituting

$$B(x, n+1) = \frac{\Gamma(x)\, n!}{\Gamma(x+n+1)}$$

(26) yields

$$x^{-p} = \sum_{n=p-1}^{\infty} \frac{S^{n}_{n+1-p}}{x(x+1)\ldots(x+n)} \tag{27}$$

For $p = 1$ the right-hand side reduces to $1/x$ because $S^0_0 = 1$ and $S^n_n = 0$ for $n \geqq 1$. Therefore (27) is effective for $p \geqq 2$.

Let ρ be a positive number and transform (22) as follows:

$$B(x, n+1) = \int_o^1 z^{x+\rho-1}\; z^{-\rho}\; (1-z)^n \, dz$$

$$= \int_o^1 z^{x+\rho-1}\; (1-(1-z))^{-\rho}\; (1-z)^n \, dz$$

$$= \int_o^1 z^{x+\rho-1} \sum_{s=o}^{\infty} (-1)^s \binom{-\rho}{s} (1-z)^{s+n} \, dz$$

$$= \sum_{s=o}^{\infty} \binom{\rho+s-1}{s} \int_o^1 z^{x+\rho-1}\; (1-z)^{s+n} \, dz$$

$$= \sum_{s=o}^{\infty} \binom{\rho+s-1}{s} \frac{(n+s)!}{(x+\rho)\ldots(x+\rho+r+s)}$$

Substituting this into (26), we find

$$x^{-p} = \sum_{n=p-1}^{\infty} \frac{C_p^n \, n!}{(x+\rho)\dots(x+\rho+n)} \qquad (28)$$

where

$$c_p^n = \sum_{r=p-1}^{n} \frac{1}{r!} \; S_{r-p+1}^{r} \binom{\rho+n-r-1}{n-r}$$

For p = 1, (28) simplifies considerably. In the following we derive the same result by a simpler method. We now start from the formula

$$\frac{1}{x} = \int_0^1 z^{x-1} \, dz$$

which transforms

$$= \int_0^1 z^{x+\rho-1} \, z^{-\rho} \, dz$$

$$= \int_0^1 z^{x+\rho-1} \, (1-(1-z))^{-\rho} \, dz$$

$$= \sum_{s=0}^{\infty} \binom{\rho+s-1}{s} \int_0^1 z^{x+\rho-1} \, (1-z)^s \, dz$$

$$= \sum_{s=0}^{\infty} \binom{\rho+s-1}{s} B(x+\rho, \; s+1)$$

$$= \sum_{s=0}^{\infty} \binom{\rho+s-1}{s} \frac{s!}{(x+\rho)\dots(x+\rho+s)}$$

Thus we find

$$\frac{1}{x} = \frac{1}{x+\rho} + \frac{\rho}{(x+\rho)(x+\rho+1)} + \frac{\rho(\rho+1)}{(x+\rho)(x+\rho+1)(x+\rho+2)} + \dots \qquad (29)$$

By a similar method a multiplier on x can be introduced into the denominators of any of the formulas (27), (28), and (29). This is the form used by Wassow (ref. 2, Theor. 47.1). We however do not need such a generalization, and will not derive such a formula. In the following we rather derive formulas in which x is raised into a numerator.

Transform (22) as follows:

$$
\begin{aligned}
B(x, n+1) &= \int_o^1 (1-(1-z))^{x-1} (1-z)^n \, dz \\
&= \sum_{r=o}^{\infty} (-1)^r \binom{x-1}{r} \int_o^1 (1-z)^{n+r} \, dz \\
&= \sum_{r=o}^{\infty} (-1)^r \binom{x-1}{r} \frac{1}{n+r+1} \\
&= \frac{1}{n+1} + \sum_{r=1}^{\infty} \frac{(1-x)\ldots(r-x)}{r!\,(n+r+1)}
\end{aligned}
$$

Substituting this into (26), we find

$$
x^{-p} = b_o^p + \sum_{r=1}^{\infty} b_r^p \frac{(1-x)\ldots(r-x)}{r!}, \tag{30}
$$

where

$$
b_o^p = \sum_{n=p-1}^{\infty} \frac{S_{n+1-p}^n}{(n+1)!}
$$

$$
b_r^p = \sum_{n=p-1}^{\infty} \frac{S_{n+1-p}^n}{n!\,(n+r+1)}
$$

For p = 1, (30) simplifies considerably. In the following we derive the same result by a simpler method. We start from the integral

$$\frac{1}{x} = \int_o^1 z^{x-1}\, dz,$$

which transforms

$$\frac{1}{x} = \int_o^1 (1-(1-z))^{x-1}\, dz$$

$$= \sum_{r=o}^{\infty} (-1)^r \binom{x-1}{r} \int_o^1 (1-z)^r\, dz$$

$$= \sum_{r=o}^{\infty} (-1)^r \binom{x-1}{r} \frac{1}{r+1}$$

Thus we find

$$\frac{1}{x} = 1 + \frac{1-x}{2!} + \frac{(1-x)(2-x)}{3!} + \frac{(1-x)(2-x)(3-x)}{4!} + \ldots \qquad (31)$$

or, writing similary to (30),

$$\frac{1}{x} = b_o^1 + \sum_{r=1}^{\infty} b_r^1 \frac{(1-x)\ldots(r-x)}{r!}$$

where

$$b_r^1 = \frac{1}{r+1}, \qquad r \geqq 0$$

The right-hand sides of (30) and (31) are a form called by Nörlund (ref. 5) as a Newtonian interpolation series

$$\sum_{r=1}^{\infty} a_r \frac{(1-x)(2-x)\ldots(r-x)}{r!} \qquad (32)$$

The range of convergence of a Newtonian interpolation series is quite different from a power series; if the former converges at x_o, it converges, similarly to a factorial series, for $x \geqq x_o$. The strange impression one may have on reading this statement will reduce when one observes that (32) ends with a finite number of terms for any positive integer x.

The convergent coordinate λ is the least x_o such that for $x < x_o = \lambda$ the series does not converge. It is given by

$$\lambda = \limsup_{n \to \infty} \frac{\log \left| \sum_{s=o}^{n-1} a_s \right|}{\log n} \tag{33}$$

when $\lambda \geqq 0$ and by

$$\lambda = \limsup_{n \to \infty} \left(\log \left| \sum_{s=n}^{\infty} a_s \right| \right) / \log n \tag{34}$$

when $\lambda < 0$. The notation $\limsup_{n \to \infty} c_n$ means that if c_{n+1} is less than the preceding numbers $c_1, c_2, \quad , c_n$, the number c_{n+1} should be discarded in taking the limit.

The range of convergence of a factorial series

$$\sum_{s=o}^{\infty} \frac{a_s \, s!}{x(x+1)\ldots(x+s)} \tag{35}$$

is similar to (32). Its convergent coordinate λ is given by

$$\lambda = \limsup_{n \to \infty} \left(\log \left| \sum_{s=o}^{n} a_s \right| \right) / \log n \tag{36}$$

when $\sum_{s=c}^{\infty} a_s$ diverges and by

$$\lambda = \limsup_{n \to \infty} \left(\log \left| \sum_{s=n+1}^{\infty} a_s \right| \right) / \log n$$

when $\sum_{s=n+1}^{\infty} a_s$ converges. In (36) $\lambda \geqq 0$, and in (37) $\lambda \leqq 0$. The formulas of convergent coordinates in the above are listed from Nörlund (ref. 4 and 5). Borel (ref. 3) proves that the Newtonian interpolation formula converges or diverges when the corresponding factorial series converges or diverges. (Note: For clarification of the last statement I must re-check Borel (ref. 3), which was borrowed from M.I.T.). This theorem is important for substituting a Newtonian interpolation series for a factorial series.

Using (25) we can show that

$$(1-x)\ldots(r-x) = \sum_{\nu=o}^{r} (-1)^{\nu} \quad S_{r-\nu}^{r+1} \quad x^{\nu} \tag{38}$$

Use of this formula reduces (30) to

$$x^{-p} = \sum_{n=o}^{\infty} c_n^p \quad x^n , \tag{39}$$

where we can find that

$$c_n^p = (-1)^n \sum_{\ell=n+p-1}^{\infty} \frac{1}{(\ell+1)!} \sum_{r=n}^{-p+1} \binom{\ell}{r} S_{r-n}^{r+1} S_{\ell-r-p+1}^{\ell-r}$$

For p = 1, the coeficients reduce to

$$c_n^1 = (-1)^n \sum_{-n+p-1}^{\infty} \frac{!}{(\ell+1)!} S_{\ell-n}^{\ell+1}$$

Eq (39) is the formula to be used to find a power series expansion at x = 0 as the analytical continuation of the asymptotic power series solution. Coefficients c_o^p,

$$c_o^p = \sum_{=p-1}^{\infty} \frac{1}{\ell+1} \sum_{r=o}^{-p+1} \frac{1}{(\ell-r)!} S_{\ell-r-p+1}^{\ell-r}$$

diverge, although the divergency is not obvious except c_0^1,

$$c_o^1 = \sum_{\ell=0}^{\infty} \frac{1}{\ell+1} \quad .$$

If the analytical continuation to x = 0 of an asymptotic power series is convergent, we must be able to rearrange the order of summation to find a finite value at x = 0. (The numerical computation was not successful, however, in this case because of the slow convergence.) Formulas (30) and (31) are interesting in itself because it shows close relationship between a factorial series and a Newtonian interpolation series. Moreover (38) is useful for finding the successive derivatives of (1-x)...(r-x).

5. Determination of Constant a and the Ratio A:B

Constant a and the ratio A:B in the asymptotic power series solution are found from the boundary conditions at $x = x_o$. The asymptotic power series diverges when n is increased indefinitely for a fixed x. It must be converted to a factorial series or a Newtonian interpolation series whose convergences are guaranteed. Such a series can be found in our problem by substituting a suitable expression of x^{-p} in the previous section into the asymptotic power series solution and the order of summations is changed accordingly. Substitution, however, may be postponed until algebraic manipulations for solving the simultaneous equations for constant a and the ratio A:B are finished because the algebraic manipulations are admissible to an asymptotic power series.

From (12) we find

$$\frac{dw}{dx} = Ce^{-x/\sqrt{2}} \sum_{n=o}^{\infty} x^{-\frac{1}{2}-n} \exp\left[\pm i\left(\frac{x}{\sqrt{2}} + \frac{(n-1)\pi}{4}\right)\right] \sum_{r=o}^{n} \gamma_r^n a^r,$$

where

$$\gamma_o^o = -1, \quad \gamma_r^o = 0 \qquad \text{for } r \geqq 1$$

$$\gamma_n^n = -\beta_n^n$$

$$\gamma_r^n = -\beta_r^n + \left(\frac{1}{2} - n\right) \beta_r^{n-1} \quad \text{for } 0 \leqq r \leqq n-1$$

In real form

$$\frac{dw}{dx} = A\frac{dw_1}{dx} + B\frac{dw_2}{dx}\ , \tag{40}$$

where

$$\frac{dw_1}{dx} = -e^{-x/\sqrt{2}} \sum_{n=o}^{\infty} x^{-n-\frac{1}{2}} \cos\left(\frac{x}{\sqrt{2}} + \frac{(n-1)\pi}{4}\right) \sum_{r=o}^{n} \gamma_r^n a^r$$

(41)

$$\frac{dw_2}{dx} = -e^{-x/\sqrt{2}} \sum_{n=o}^{\infty} x^{-n-\frac{1}{2}} \sin\left(\frac{x}{\sqrt{2}} + \frac{(n-1)\pi}{4}\right) \sum_{r=o}^{n} \gamma_r^n a^r$$

Or exchanging the order of summations,

$$\frac{dw_1}{dx} = -e^{-x/\sqrt{2}} \sum_{r=o}^{\infty} a^r \sum_{n=r}^{\infty} \gamma_r^n x^{-n-\frac{1}{2}} \cos\left(\frac{x}{\sqrt{2}} + \frac{(n-1)\pi}{4}\right)$$

(42)

$$\frac{dw_2}{dx} = -e^{-x/\sqrt{2}} \sum_{r=o}^{\infty} a^r \sum_{n=r}^{\infty} \gamma_r^n x^{-n-\frac{1}{2}} \sin\left(\frac{x}{\sqrt{2}} + \frac{(n-1)\pi}{4}\right)$$

Use of (13) and (40) in the condition at $x = x_o$ yields

$$\frac{A}{B} = - \frac{w_2(x_o)}{w_1(x_o)} = - \left(\frac{dw_2}{dx}\right)_{x_o} \Bigg/ \left(\frac{dw_1}{dx}\right)_{x_o} \qquad (43)$$

Eliminating the ratio A:B, the equation for determination of constant a is found

$$\sum_{t=o}^{\infty} a^t \Gamma_t(x_o) = o, \qquad (44)$$

where

$$F_t(x_o) = \sum_{p=t}^{\infty} x_o^{-p} \quad L_t^p \tag{45}$$

and

$$L_t^p = \sum_{r=o}^{t} \sum_{n=r}^{r+p-t} \gamma_r^n \, \beta_{t-r}^{p-n} \quad \sin \frac{(p-2n+1)\pi}{4} \tag{46}$$

When p=t constants L_t^p approach zero as p=t increases, and for a fixed t constant L_t^p increases rapidly in their absolute values as p increases indefinitely. The wide range of values covered by constants L_t^p was one of the causes of the computational difficulty.

Using (27), functions $F_t(x_o)$ are expressed as convergent factorial series,

$$\left.\begin{aligned}
F_o(x_o) &= L_o^o + \frac{L_o^1}{x_o} + \sum_{r=1}^{\infty} \frac{1}{x_o(x_o+1)\dots(x_o+r)} \sum_{p=2}^{r+1} L_o^p \quad S_{r-p+1}^r \\
F_1(x_o) &= \frac{L_1^1}{x_o} + \sum_{r=1}^{\infty} \frac{1}{x_o(x_o+1)\dots(x_o+r)} \sum_{p=2}^{r+1} L_1^p \quad S_{r-p+1}^r \\
&\text{and} \\
F_t(x_o) &= \sum_{r=t-1}^{\infty} \frac{1}{x_o(x_o+1)\dots(x+r)} \sum_{p=t}^{r+1} L_t^p \quad S_{r-p+1}^r
\end{aligned}\right\} \tag{47}$$

for $t \geqq 2$. The convergence of these factorial series is guaranteed because w and dw/dx expressed as factorial series converge.

When x_o approaches zero as the limit, we replace $F_t(x_o)$ with $G_t(x_o) = x_o F_t(x_o)$, whose values $G_t(o)$ at $x = 0$ for $t \geq 0$ are finite:

$$G_o(0) = L_o^1 + \sum_{r=1}^{\infty} \frac{1}{r!} \sum_{p=2}^{r+1} L_o^p \; S_{r-p+1}^r$$

$$G_1(0) = L_1^1 + \sum_{r=1}^{\infty} \frac{1}{r!} \sum_{p=2}^{r+1} L_1^p \; S_{r-p+1}^r \tag{48}$$

$$G_t(0) = \sum_{r=t-1}^{\infty} \frac{1}{r!} \sum_{p=t}^{r+1} L_t^p \; S_{r-p+1}^r$$

Therefore constant a tends to a certain value when x_o approaches zero as the limit.

When x_o increases indefinitely we may use the asymptotic power series (45) for evaluating constant a from (44). We can show in this case that

$$k = a/x_o \tag{50}$$

approaches a certain constant as the limit when x_o increases indefinitely, where k is the root of

$$\sum_{k=o}^{\infty} L_n^n \; k^n = 0 \tag{51}$$

It is noted that the asymptotic power series (45) cannot be used at a finite x_o for evaluating constant a.

The series (48) converged rapidly and the computation of constant a in the neighborhood of $x_o = 0$ was carried out. The result is shown in Table 1. The series (51), however, did not numerically converge, because

the number k was so large that the series could not converge even in the utmost capability of the DTSS.

The most serious limitation of the DTSS was the narrow range (approximately from 1E-39 to 1E+38) of the magnitude of the numbers that it can handle. The graph $y = \sum_{k=o}^{\infty} L_n^n X^n$ touches the x axis extremely gently and the root k of equation (51) could not be easily discovered.

In order to determine A/B in (43) computation of the middle term, $-w_2(x_o)/w_1(x_o)$, and the right term, $-\left(\frac{dw_1}{dx}\right)_{x_o} \Big/ \left(\frac{dw_1}{dx}\right)_{x_o}$, was carried out through 25 terms. It was found out that the two series did not converge sufficiently for all values of x_o to test whether or not they were equal (Table 2).

Further computations were not attempted.

The language used on the DTSS was the BASIC.

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Table 1. Values of a as function of x_o.

x_o	a	$k = a/x_o$
0	3.18364	
1E-6	3.18364	
1E-5	3.18368	
1E-4	3.18405	
1E-3	3.18775	
1E-2	3.22414	
1E-1	11.1805	
1	17.1648	17.16
2	48.9837	24.49
3	68.3702	22.79
4	84.172	21.04
5	99.2279	19.84
6	113.586	18.93
7	127.65	18.24
8	141.59	17.69
9	155.48	17.28
10	fails to converge	

Table 2. Selected values of the Terms in Equation (43).

x_o	a		$-\frac{w_2(x_o)}{w_1(x_o)}$	$-\left(\frac{dw_2}{dx}\right)_{x_o} \Big/ \left(\frac{dw_1}{dx}\right)_{x_o}$
1	17.1681	S_{25}	0.890623	-0.625972
		S_{24}	1.06104	-0.716557
0.001	3.18775	S_{25}	-3.20247	-5.77656
		S_{24}	-3.73226	-6.23476

Note: S_{25} and S_{24} are the sums of 25 terms and 24 terms of the series under consideration. They show that the series are yet far from convergence.